# First Midterm: Preparation

MA441: Algebraic Structures I

6 October 2003

1) Definitions (two from this list)

1. Define a **group**.
2. Define what it means for a subset of a group to be a **subgroup**.
3. Define what it means for a group to be **cyclic**.
4. Define $U(n)$, the group of units modulo $n$. Specify the elements and the group operation.
5. Define $\mathrm{GL}(2, \mathbb{R})$. Specify the elements and the group operation. (You don't need to write any formulas.)
6. Define $|a|$, the **order** of an element of a group.

2) Examples (two from this list)
Give an example of

1. a noncyclic group of order 4.
2. a nonabelian group of order 10.
3. an element of order 2 in $\mathrm{GL}(2, \mathbb{R})$.
4. an infinite nonabelian group.
5. an infinite abelian group
6. an abelian subgroup of a nonabelian group.

7. a group with exactly five subgroups (including the trivial subgroup and itself).

3) Fill in the blanks (five from this list)

1. The order of 4 in $U(7)$ is __.
2. If $\alpha = (1, 3, 2)$, a permutation on $\{1, 2, 3\}$, then $\alpha^2 =$ __.
3. The order of the group $U(13)$ is __.
4. If $x \in G$, $x \neq e$, and $x^{10} = x^{35} = e$, then $|x| =$ __.
5. In the group $D_4$, let $R$ denote rotation by 90 degrees counterclockwise, and let $F$ denote a flip about the vertical. Written in the form $F^i R^j$, the element $RF$ equals __.
6. A complete list of all generators of $\mathbb{Z}/6\mathbb{Z}$ is __.
7. True or False: Every abelian group is cyclic. __
8. True or False: $U(5)$ is a subgroup of $\mathbb{Z}/5\mathbb{Z}$. __
9. True or False: Let $a$ be an element of $G$. The inverse of $a$ is always $1/a$. __

4) Euclidean algorithm (similar but with different numbers)

1. Use the Euclidean Algorithm to express $\gcd(37, 5)$ as an integer linear combination of 37 and 5. Show your work.
2. Find the multiplicative inverse of 5 in $U(37)$. Show your work.

5) Permutations (similar but with different numbers)
Consider permutations of the set $\{1, 2, 3, 4, 5, 6\}$.

1. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 4 & 2 & 5 & 1 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 6 & 3 & 1 & 4 \end{pmatrix}.$$

   Write $\alpha$ and $\beta$ in cycle notation.

2. Compute the composition $\alpha\beta$ and write in cycle notation. Write $\alpha^{-1}$ in cycle notation.

6–7) Short proofs (two from this list, each graded as a separate problem) Explain your arguments with clear and coherent sentences.

1. Prove that if $|a| = k$, then $\langle a \rangle = \{e, a, a^2, \dots, a^{k-1}\}$.
2. Let $a$ be an element of a group $G$. Prove that $\langle a \rangle = \{a^n : n \in \mathbb{Z}\}$ is a subgroup of $G$.
3. Define the center $Z(G)$ of a group $G$. Prove that $Z(G)$ is a subgroup of $G$.
4. Prove that the inverse of an element $a$ in a group $G$ is unique.
5. Let $H$ be a nonempty finite subset of a group $G$. Then $H$ is a subgroup of $G$ if $H$ is closed under the operation of $G$.
6. An assigned homework problem. (Homeworks 1–3 only)