# Final Exam

MA441: Algebraic Structures I

17 December 2003

All questions are worth ten points, unless otherwise indicated. In addition to the questions, there will be an additional ten points to be awarded for style and clarity in writing. Please sign the honor code pledge on the back of your exam book.

1) Definitions (20 points)

1. Given a subgroup $H \lhd G$, define the quotient group $G/H$. (Describe the set and the group operation.)
2. Given a permutation group $G < S_n$ acting on the set $\{1, 2, \ldots, n\}$, define the stabilizer $\mathrm{Stab}_G(i)$.
3. Given an element $a \in G$, define the centralizer $C(a)$.
4. Given $a \in G$, define the conjugacy class $\mathrm{cl}(a)$.

2) Fill in the blanks or answer True/False (five from this list)

1. True or False: $(1234)(4567) \in A_7$. __
2. True or False: $\langle (14) \rangle$ is a normal subgroup of $S_4$. __
3. True or False: If 7 divides $|G|$, then $G$ has an element of order 7. __
4. True or False: For every positive integer $n$, $\mathrm{Aut}(\mathbb{Z}_n) \approx U(n)$. __
5. True or False: Let $G$ be a cyclic group of order $n$. If $k|n$, then there is an $H < G$ such that $H$ has order $k$. __

3) Let $H$ be a nonempty finite subset of a group $G$. Prove that $H$ is a subgroup of $G$ if $H$ is closed under the operation of $G$.

4) Use Lagrange's Theorem to prove Fermat's Little Theorem: for every integer $a$ and every prime $p$, $a^p \equiv a \pmod{p}$.

5) Cosets

1. Given a subgroup $H < G$ and any $a, b \in G$, prove that either $aH = bH$ or $aH \cap bH = \emptyset$, i.e., $aH$ and $bH$ are disjoint.

2. Given a subgroup $H < G$ and any $a \in G$, prove that $aH < G$ iff $a \in H$.

6) Homomorphisms. Let $\phi : G_1 \to G_2$ be a homomorphism, and let $H < G_1$.

1. Prove that $\phi(H)$ is a subgroup of $G_2$.

2. Prove that if $H \lhd G_1$ then $\phi(H) \lhd \phi(G_1)$.

7) First Isomorphism Theorem. Let $\phi : G \to H$ be a homomorphism of groups and let $K = \operatorname{Ker} \phi$. Let $\psi : G/K \to H$ be the correspondence that sends $gK \mapsto \phi(g)$.

1. Prove that if $K = \{e\}$, then $\phi$ is one-to-one.

2. Show that $\psi$ is well-defined. Prove that for any $x, y \in G$ such that $xK = yK$, we have $\psi(xK) = \psi(yK)$.

8) Euclidean Algorithm

1. Use the Euclidean Algorithm to express $\gcd(13, 28)$ as an integer linear combination of 13 and 28. Show all work.

2. Find the inverse of 13 in $U(28)$.

9) Prove that if $|a| = k$, then $\langle a \rangle = \{e, a, a^2, \ldots, a^{k-1}\}$.

10) Let $G$ be the group $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ under addition, and let $H$ be the group

$$H = \left\{ \left[ \begin{array}{cc} a & 2b \\ b & a \end{array} \right] \middle| \; a, b \in \mathbb{Q} \right\}$$

under addition.

Show that $G$ and $H$ are isomorphic under addition.

11) Suppose that $|x| = n$. Find a necessary and sufficient condition on $r$ and $s$ such that $\langle x^r \rangle \subseteq \langle x^s \rangle$. Justify your answer.

12) (for 10 points of extra credit) Prove Lagrange's Theorem. You may cite basic properties of cosets, such as those listed in Gallian's Lemma in Chapter 7, if you state them accurately.